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On Representations of Artin Groups and the Tits Conjecture

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1. INTRODUCTION

The n-string braid group B_n, having standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and presentation

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1, \ldots, n-1 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| > 1 \end{aligned}$$

appears in many different contexts [Bi]. It occurs as the mapping class group of the n-punctured disc, where each generator σ_i is a half-twist. It also occurs as a subgroup of $\mathrm{Aut}(F_n)$, the automorphism group of the free group F_n of rank n. It also appears as a special kind of Artin group; here an Artin group $A(M)$ with $n \times n$ Coxeter matrix $M = (m_{ij})$ ($m_{ij} = m_{ji} \in \mathbb{Z}^{\geq 2} \cup \{\infty\}$ for $i \neq j$ and $m_{ii} = 1$) is a group with generating set $\{a_1, \ldots, a_n\}$ and for each $i \neq j$ a relation

$$a_i a_j a_i \ldots = a_j a_i a_j \ldots,$$

where both sides are words of length m_{ij}. We will say that $A(M)$ is of *small type* if $m_{ij} \leq 3$ for all i, j. One notes that if $m_{ij} = 2$, then a_i and a_j commute and thus so do a_i^2 and a_j^2. The Tits conjecture [A-S] referred to in the title says that for an arbitrary Artin group the only relations between the a_i^2 are these obvious commutator relations. Various cases of this conjecture have been proved [A-S, P], including Artin groups of *extra-large type* ($m_{ij} \geq 4$ for all $i \neq j$) and for B_4 and B_5 [D-L-S]. Another kind of Artin group is the graph group: if Γ is a graph we let $\langle \Gamma \rangle$ be the group with generators corresponding to the vertices of Γ and relators $\{xy = yx \mid x \text{ and } y \text{ are non-adjacent vertices}\}$. The Tits conjecture then says that $\langle a_1^2, \ldots, a_n^2 \rangle$ is a graph group.

In this paper we show that we can represent Artin groups $A(M)$ of small type as groups of automorphisms of the free group F_n. We then

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show how each such group gives rise to an action of $A(M)$ as automorphisms of a free polynomial algebra. This generalises a certain representation of B_n as a group of automorphisms of a polynomial algebra that we constructed in [Hu1]. We use this representation to prove the Tits conjecture for the braid groups and for certain other Artin groups of small type. We do this by obtaining a linear representation of the subgroup $\langle a_1^2, \ldots, a_n^2 \rangle$ of $A(M)$ by linearising the action on the polynomial algebra at a fixed point of $\langle a_1^2, \ldots, a_n^2 \rangle$ using the Jacobian. In fact for $i = 1, \ldots, n$, each a_i^2 is represented as a transvection and in the course of the proof we show that any graph group can be represented as a linear group over $\mathbb{C}$, where the generators $a_1^2, \ldots, a_n^2$ are represented as transvections.

We now give a graphical description of Artin groups of small type. Let Γ be a finite graph with vertices $v_1, \ldots, v_n$ and let $E = E(\Gamma)$ be the set of edges of Γ. For each $e = \{v_i, v_j\} \in E$ let $\sigma(e) = \sigma_{ij}$ be a generator for a group $A(\Gamma)$. For every pair e_1, e_2 of adjacent edges in Γ we have a relator

$$\sigma(e_1)\sigma(e_2)\sigma(e_1) = \sigma(e_2)\sigma(e_1)\sigma(e_2),$$

and for every pair e_1, e_2 of non-adjacent edges in Γ we have a relator

$$\sigma(e_1)\sigma(e_2) = \sigma(e_2)\sigma(e_1).$$

Then clearly $A(\Gamma)$ is an Artin group of small type, but not every Artin group $A(M)$ of small type is of this form. Note that if Γ is not connected, then $A(\Gamma)$ is a direct product of Artin groups of small type. Thus we assume in what follows that Γ is connected. The result that we prove is

THEOREM 1.1. *If Γ is a finite, connected, bipartite graph which contains no 4-gon subgraphs, then the subgroup $\langle \sigma(e)^2 | e \in E \rangle$ of $A(\Gamma)$ has the presentation $\langle \sigma(e)^2 | \sigma(e_1)^2\sigma(e_2)^2 = \sigma(e_2)^2\sigma(e_1)^2$ if e_1, e_2 are non-adjacent edges in $\Gamma \rangle$.*

In Section 6 we will also prove a generalisation of the Tits conjecture in the case of braid groups.

2. THE REPRESENTATION INTO $\mathrm{Aut}(F_n)$

We now explain how to obtain a representation of $A(\Gamma)$ into $\mathrm{Aut}(F_n)$, the automorphism group of a free group F_n. Let $\Phi\colon \Gamma \to F_{g,1}$ be a proper embedding of Γ into an oriented surface $F_{g,1}$ of genus g with a single boundary component. See [B-C-L] for the existence of such an embedding. Let

$$F_{g,1}^* = F_{g,1} \setminus \{\Phi\{v_1, \ldots, v_n\} \cup \{p_1, \ldots, p_k\}\},$$

where $p_1, \ldots, p_k$ are points of $F_{g,1}$, one in each component of $F_{g,1} \setminus \Phi\{\Gamma\}$ that does not contain $\partial F_{g,1}$ the boundary of $F_{g,1}$. We call $\Phi\{v_1, \ldots, v_n\} \cup \{p_1, \ldots, p_k\}$ the *punctures* of $F^*_{g,1}$. Let $p \in \partial F_{g,1}$. The mapping class group $\mathbf{M(F^*_{g,1})}$ of $F^*_{g,1}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $F^*_{g,1}$ which fix $\partial F^*_{g,1}$ and permute the punctures. Now via Φ each edge e in $E(\Gamma)$ determines an open arc e^* in $F^*_{g,1}$ which connects two punctures of $F^*_{g,1}$. Let $h(e) \in M(F^*_{g,1})$ be the half-twist [Bi, p.33] in a small tubular neighbourhood of this arc relative to the fixed orientation of the surface. Let $H(\Gamma)$ be the subgroup $\langle h(e) | e \in E(\Gamma)\rangle$ of $M(F^*_{g,1})$. Note that the square of a half-twist is a Dehn twist [Bi] about the simple closed curve which is the boundary of a closed tubular neighbourhood of e^*. Since $h(e)$ is a half-twist we see that if e_1, e_2 are adjacent edges in Γ, then

$$h(e_1)h(e_2)h(e_1) = h(e_2)h(e_1)h(e_2),$$

while if e_1, e_2 are non-adjacent edges in Γ, then the tubular neighbourhoods of e_1 and e_2 are disjoint and so

$$h(e_1)h(e_2) = h(e_2)h(e_1).$$

We thus obtain a homomorphism ψ: $A(\Gamma) \to H(\Gamma)$ induced by setting $\psi(\sigma(e)) = h(e)$ for each $e \in E(\Gamma)$. Now $H(\Gamma)$ fixes the point p and so there is an induced map ψ^*: $A(\Gamma) \to H(\Gamma) \to \mathrm{Aut}(\pi_1(F^*_{g,1}, p)) = \mathrm{Aut}(F_m)$, where $\pi_1(F^*_{g,1}, p)$ is a free group of rank m.

We next investigate the nature of the automorphisms $\psi^*(\sigma(e_{ij})) = \tau_{ij}$. Now we can choose the free generators for $\pi_1(F^*_{g,1}, p)$ in the following way: first choose a set of $2g$ free generators for $\pi_1(F_{g,1}, p) \cong F_{2g}$ missing the punctures and such that $F_{g,1} \setminus (c_1 \cup \cdots \cup c_{2g})$ is connected and lift them to elements $c_1, \ldots, c_{2g}$ of $\pi_1(F^*_{g,1}, p)$. Now for each puncture $\Phi(v_i)$, $i = 1, \ldots, n$, or p_j, $j = 1, \ldots, k$, we choose arcs ρ_i, $i = 1, \ldots, n + k$, in $F^*_{g,1}$ from p to the puncture which are disjoint, except at p and let x_i, $i = 1, \ldots, n + k$, be homotopically non-trivial simple closed arcs based at p, contained in a small tubular neighbourhood of ρ_i and oriented positively. Then $c_1, \ldots, c_{2g}, x_1, \ldots, x_{n+k}$ generate $\pi_1(F^*_{g,1}, p)$. Figure 1 shows c_1, c_2, and some of $x_1, \ldots, x_{n+k}$ for the case $\Gamma = K_5$, the complete graph on five vertices where $n = 5$, $k = 4$, and $g = 1$.

It is now easily seen that $\tau_{ij}(x_h)$ is a conjugate of x_h if $h \neq i, j$, that $\tau_{ij}(x_i)$ is a conjugate of x_j, and that $\tau_{ij}(x_j)$ is a conjugate of x_i. Further, for $h = 1, \ldots, 2g$, $\tau_{ij}(c_h)c_h^{-1}$ is in the kernel of the map $\pi_1(F^*_{g,1}, p) \to \pi_1(F_{g,1}, p)$. Now let ξ: $F_m \to F_{n+k} = \langle x_1, \ldots, x_{n+k}\rangle$ be the epimorphism induced by sending $c_1, \ldots, c_{2g}$ to the identity. Then by the above ξ

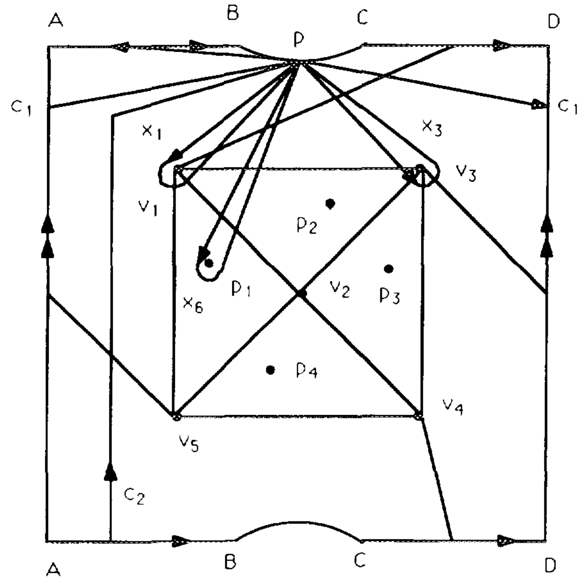

FIGURE 1

induces a map ξ': $\mathrm{Aut}(F_m) \to \mathrm{Aut}(F_{n+k})$ and so we now have a homomorphism,

$$\xi^* = \xi' \circ \psi^*: A(\Gamma) \to \mathrm{Aut}(F_{n+k}),$$

which, by the above remarks, has its image contained in the subgroup $\hat{\mathrm{H}}(n+k)$ of $\mathrm{Aut}(F_{n+k})$ consisting of all the automorphisms of $F_{n+k} = \langle x_1, \ldots, x_{n+k} \rangle$ which send each generator x_i to a conjugate of some x_j, where $i \to j$ is a permutation of $1, \ldots, n+k$. This gives the description of the action of $A(\Gamma)$ as elements of $\hat{\mathrm{H}}(n+k)$.

For example, if Γ is a line graph with n vertices, then $g = 0$, $k = 0$, $m = n$, $A(\Gamma) = B_n$, and we obtain the well-known (faithful) representation $B_n \to \mathrm{Aut}(F_n)$ studied by Birman *et al.* [Bi, p. 30]. If Γ is a polygonal graph with n vertices and n edges, then $g = 0$, $m = n + 1$, and $A(\Gamma)$ is the so-called group of circular braids studied by Johnson *et al.* [A-J]. The case where Γ is a star graph has been studied by Napthine [N]. The faithfulness of ξ^* in most cases of interest is given by

PROPOSITION 2.1. *The representation ξ^*: $A(\Gamma) \to \mathrm{Aut}(F_{n+k})$ is not faithful for any bipartite graph Γ containing a vertex of valence* 3 *or more.*

Proof. Suppose that Γ has four vertices $v_1, \ldots, v_4$, where v_1, v_2, v_3 are adjacent to v_4. Then v_1, v_2, v_3 are not adjacent since Γ is bipartite. Let Γ'

be the full subgraph on $v_1, \ldots, v_4$. Then a tubular neighbourhood of the image of Γ' is planar and we can choose generators $x_1, \ldots, x_{n+k}$ such that the action of the three edges on $\langle x_1, \ldots, x_{n+k} \rangle$ is the same as $\sigma_1, \sigma_2, \sigma_2\sigma_3\sigma_2^{-1}$, where $\sigma_1, \sigma_2, \sigma_3$ are the standard generators of B_4. Thus the image of $A(\Gamma')$ under ξ^* is B_4. However, one can show that $A(\Gamma')$ is not isomorphic to B_4 by, for example, using ones favourite group-theory package to count the number of subgroups of index 5 or less in B_4 and in $A(\Gamma')$. ∎

By the above result the only bipartite graphs Γ for which ξ^* could be faithful are the braid groups, for which ξ^* is faithful, and the circular braid groups.

We next give some background on the subgroups $\hat{H}(n) < \mathrm{Aut}(\langle x_1, \ldots, x_n \rangle)$. These are generated [Hu2] by the permutation automorphisms π_{ij}: $\pi_{ij}(\mathbf{x}_k) = \mathbf{x}_k$ if $k \neq i, j$, $\pi_{ij}(\mathbf{x}_i) = \mathbf{x}_j$ and $\pi_{ij}(\mathbf{x}_j) = \mathbf{x}_i$; and by the elements

$$t_{ij}: t_{ij}(\mathbf{x}_k) = \mathbf{x}_k \qquad \text{if } k \neq j,\ t_{ij}(\mathbf{x}_j) = \mathbf{x}_i\mathbf{x}_j\mathbf{x}_i^{-1}.$$

There is an obvious split exact sequence

$$1 \to H(n) \to \hat{H}(n) \to S_n \to 1,$$

where S_n is the symmetric group. A presentation for $H(n)$, and so of $\hat{H}(n)$, can be obtained from the paper of McCool [Mc].

Let $\mathrm{Inn}(n)$ be the subgroup of inner automorphisms of $\mathrm{Aut}(F_n)$ and let $\hat{H}(n)^*$ be the quotient $\mathrm{Aut}(F_n)/\mathrm{Inn}(n)$. In [Hu1] we showed that $H(n)^*$ has no non-trivial torsion elements. In [Co] Collins shows that $H(n)^*$ acts freely with finite quotient on a contractible space Σ of dimension $n-2$. We thus have

THEOREM 2.2. *Σ is an Eilenberg–Maclane space of type $(H(n)^*, 1)$.*

3. The Representation of $\hat{H}(n)$ as Automorphisms of a Polynomial Algebra

We now describe the action of $A(\Gamma)$ as automorphisms of a polynomial algebra. This is motivated by the results of [Hu1] concerning transvections. Let R be a commutative ring with identity.

Let $\mathrm{SL}(n, R)$ denote the group of $n \times n$ matrices of determinant 1 over R. A *transvection* in $\mathrm{SL}(n, R)$ is an element T which can be specified as a (non-unique) pair (ψ, d), where $\psi \in (R^n)^*$, the dual space of R^n, $d \in R^n$, $\psi(d) = 0$, and for all $x \in R^n$ we have

$$T(x) = x + \psi(x)d.$$

Let $S = \{T_1, \dots, T_n\}$ be a set of transvections in $\mathrm{SL}(n, R)$, where $T_1 = (\psi_1, d_1), \dots, T_n = (\psi_n, d_n)$ and let $\langle S \rangle$ denote the subgroup of $\mathrm{SL}(n, R)$ generated by $T_1, \dots, T_n$. Associate to the set S the $n \times n$ matrix $M(S) = (\psi_i(d_j))$; this is a conjugacy invariant of S. Moreover, if $M(S)$ is non-degenerate, then the matrix $M(S)$ completely determines the subgroup $\langle S \rangle$ up to conjugacy. For S as above we let $\Gamma(S)$ be the graph with vertices $T_1, \dots, T_n$ and an edge between T_i and T_j if they do not commute. Now the group $\hat{\mathrm{H}}(n)$ acts on ordered sets of transvections, and the corresponding action on the matrices $M(S)$ is given by the next result.

LEMMA 3.1. [Hu1]. (i) *Let $T = (\psi, d)$, $U = (\mu, e)$ be transvections. Then for all $\lambda \in \mathbb{Z}$ we have*

$$U^\lambda T U^{-\lambda} = (\psi - \lambda\psi(e)\mu, U^\lambda(d)).$$

(ii) *Let $S = \{T_1, \dots, T_n\}$ be a set of transvections as above and let $M(S) = (a_{rs})$. Then the generators π_{ij} act on the a_{rs} by permutation of subscripts and for $\lambda \in \mathbb{Z}$, $t_{kj}^\lambda(M(S)) = (b_{rs})$, where*

$$\begin{aligned} b_{ij} &= a_{ij} + \lambda a_{ik} a_{kj}, && \text{if } i \neq j; \\ b_{ji} &= a_{ji} - \lambda a_{jk} a_{ki}, && \text{if } i \neq j; \\ b_{hi} &= a_{hi} && \text{if } h \neq j, i \neq j. \end{aligned}$$

If $S = \{T_1, \dots, T_n\}$ with $T_i = (\psi_i, d_i)$ for $i = 1, \dots, n$, and $d_1, \dots, d_n$ is a basis, then relative to this basis T_i is $I_n + F_i M(S)$, where F_i is the diagonal matrix with 1 in the (ii) position and 0's elsewhere.

If we let $\mathbb{Z}[a_{ij}]$ denote the polynomial algebra

$$\mathbb{Z}[a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{23}, \dots, a_{2n}, \dots, a_{n1}, a_{n2}, \dots, a_{nn-1}],$$

then the group $\hat{\mathrm{H}}(n)$ acts on $\mathbb{Z}[a_{ij}]$ as described in Lemma 3.1. As in [Hu1] we obtain a linear representation of a subgroup G of $\hat{\mathrm{H}}(n)$ by thinking of $\hat{\mathrm{H}}(n)$ as acting on $\mathbb{C}^{n^2-n}$, where points of $\mathbb{C}^{n^2-n}$ are given as $(n^2 - n)$-tuples in the order

$$a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{23}, \dots, a_{2n}, \dots, a_{n1}, a_{n2}, \dots, a_{nn-1}.$$

We refer to the a_{ij} as the *coordinates* of $\mathbb{C}^{n^2-n}$. A linear representation of G then comes from linearising the action of G at a fixed point P for the action of G; the chain rule then shows that we get a linear representation $\Phi_P\colon G \to GL(n^2 - n, \mathbb{Z}[a_{ij}])$ given by the Jacobian at P.

It so happens that the representations of interest to us are easier to understand in the case where $F_{g,1} \setminus \Phi(\Gamma)$ is connected, i.e., when $k = 0$. The next result shows that we can always guarantee this situation.

PROPOSITION 3.2. *Every connected graph Γ has an embedding Φ: $\Gamma \to F_{g,1}$ into an oriented surface $F_{g,1}$ of genus g with a single boundary component such that $F_{g,1} \setminus \Phi(\Gamma)$ is connected.*

Proof. Let Φ': $\Gamma \to F_{g',1}$ be any embedding of Γ into an oriented surface $F_{g',1}$ of genus g' with a single boundary component [B-C-L]. If $F_{g',1} \setminus \Phi'(\Gamma)$ is not connected, then we use an orientable 2-handle disjoint from $\Phi'(\Gamma)$ to connect two components of $F_{g',1} \setminus \Phi'(\Gamma)$. This reduces the number of components of $F_{g',1} \setminus \Phi'(\Gamma)$ and continuing in this way gives the result. ∎

A *cut-system* for the generating set $c_1, \dots, c_{2g}, x_1, \dots, x_{n+k}$ for $\pi_1(F^*_{g,1}, p)$ chosen above is a set $\{c_1^*, \dots, c_{2g}^*, x_1^*, \dots, x_{n+k}^*\}$ of oriented simple arcs, some of which may be closed, such that

$$F^*_{g,1} \setminus \left(c_1^* \cup \cdots \cup c_{2g}^* \cup x_1^* \cup \cdots \cup x_{n+k}^*\right)$$

is connected and such that c_i crosses c_i^* once in the positive sense for $i = 1, \dots, 2g$, x_i^* has one end on a puncture and the other on $\partial F^*_{g,1}$, and x_i crosses x_i^* once in the positive sense for $i = 1, \dots, n+k$. It is always possible to find a cut-system for the generators $c_1, \dots, c_{2g}, x_1, \dots, x_{n+k}$. In the situation where $F_{g,1} \setminus \Phi(\Gamma)$ is connected we can and will also require that $x_1^* \cup \cdots \cup x_{n+k}^*$ is a star and meets $\Phi(\Gamma)$ only at the vertices of $\Phi(\Gamma)$.

4. THE BIPARTITE CASE

We next consider the case where Γ is a bipartite graph with bipartite partition $P_1 \cup P_2$ of the vertices $V(\Gamma) = \{v_1, \dots, v_n\}$ of Γ. This includes the braid groups and all graphs whose cycles have even length. Let $d(\ ,\ )$ be the distance function in Γ. For any $k > 0$ let

$$C_k = \left\{\{i,j\} \mid v_i, v_j \in V(\Gamma),\, d(v_i, v_j) = k\right\}.$$

Let P be the point in $\mathbb{C}^{n^2-n}$ given by coordinates a_{ij}, where $a_{ij} = 0$, unless $\{i,j\} \in C_2$. Now we are only going to use the following subset of the coordinates of $\mathbb{C}^{n^2-n}$, namely those contained in the set

$$C = \left\{a_{ij} \mid \{i,j\} \in C_1,\, v_i \in P_1,\, v_j \in P_2\right\}.$$

Note that since Γ is bipartite, Γ contains no triangles, and so none of the above coordinates are zero for the point P. For notational convenience we write $C = \{c_1, \ldots, c_q\}$.

LEMMA 4.1. *Suppose that $F_{g,1} \setminus \Phi(\Gamma)$ is connected. Then relative to the above chosen coordinates each generator $\sigma(e_{ij})$ of $A(\Gamma)$ is represented by the Jacobian at the point P as a transvection T_{ij}. Moreover, $M(\{T_{rs}\}) = (b_{ij})$, where if $c_i = a_{rs}$ and $c_j = a_{uv}$; then*

$$\begin{aligned} b_{ij} &= 0 && \text{if } r \neq u \quad \text{and} \quad s \neq v; \\ b_{ij} &= a_{vs} && \text{if } r = u; \quad \text{and} \\ b_{ij} &= a_{ru} && \text{if } s = v. \end{aligned}$$

Proof. This will need a more detailed investigation of the action of a Dehn twist about a simple closed curve surrounding two punctures. Figure 2 shows what happens in a neighbourhood of an edge $e = e_{ij}$ of Γ. Let v_i and v_j be the vertices of e, where we assume that $v_i \in P_1$. Now up to interchanging i and j there are four possibilities for the x_i, x_i^* and x_j, x_j^*. These are indicated in Fig. 3.

For the first possibility in Fig. 3 we use Fig. 2 to get the following action of $h(e)^2$ on the free group $\langle x_1, \ldots, x_n \rangle$:

$$h(e)^2(x_i) = x_j x_i x_j^{-1}; \; h(e)^2(x_j) = x_j x_i x_j x_i^{-1} x_j^{-1};$$
$$h(e)^2(x_k) = y x_k y^{-1} \quad \text{for } k \neq i, j, \text{where } y \in N_{ij}.$$

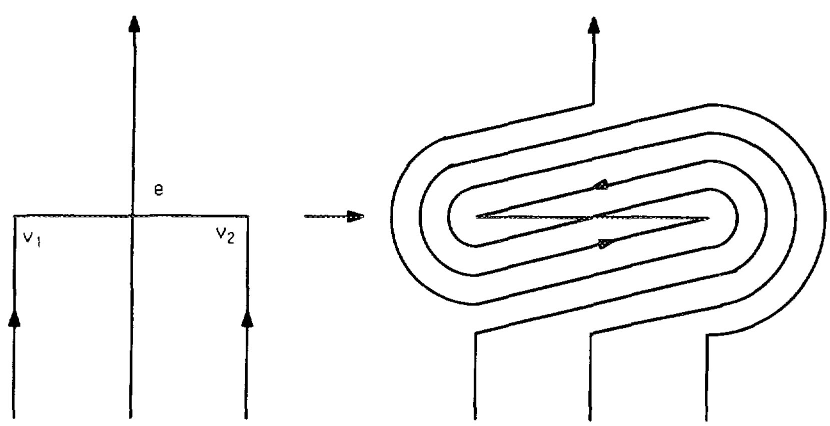

FIGURE 2

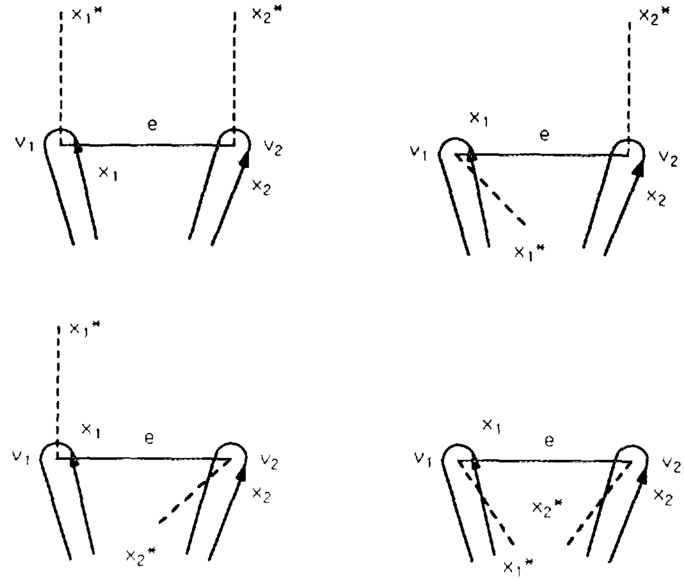

FIGURE 3

Here N_{ij} is the subgroup $\langle x_j x_i x_j^{-1} x_i^{-1}\rangle$ of $\langle x_1, \ldots, x_n\rangle$. The actions for the other cases in Fig. 3 are examined in the same way. Using Lemma 3.1 this gives the following action of $h(e)^2$ on the a_{rs}:

$$
\begin{aligned}
h(e)^2(a_{rs}) &= a_{rs} && \text{if } r, s \neq i, j;\\
h(e)^2(a_{ij}) &= a_{ij};\\
h(e)^2(a_{ri}) &= a_{ri} + a_{rj}a_{ji} && \text{if } r \neq i, j;\\
h(e)^2(a_{ir}) &= a_{ir} - a_{ij}a_{jr} && \text{if } r \neq i, j;\\
h(e)^2(a_{rj}) &= a_{rj} + a_{ri}a_{ij} + a_{rj}a_{ji}a_{ij} && \text{if } r \neq i, j;\\
h(e)^2(a_{jr}) &= a_{jr} - a_{ji}a_{ir} + a_{ji}a_{ij}a_{jr} && \text{if } r \neq i, j.
\end{aligned}
$$

Differentiating at the fixed point P described above we see that the only non-constant partial derivatives in the Jacobian are those by the coefficients a_{ij}; and we have (evaluating at the fixed point P)

$$\frac{\partial}{\partial a_{ijP}}\left(h(e_{ij})^2(a_{ir})\right) = -a_{jr} \qquad \text{if } \{j,r\} \in C_2 \text{ etc.}$$

It follows that each $h(e_{ij})^2$ in $A(\Gamma)$ is represented as a transvection T_{ij}. ∎

EXAMPLE 4.2. *If Γ is a 4-gon, then we get the following transvections*:

$$T_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_{24} & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{23} = \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_{24} & 1 \end{pmatrix},$$

$$T_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_{13} \\ 0 & 0 & 1 & -a_{42} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_{41} = \begin{pmatrix} 1 & -a_{42} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a_{31} & 0 & 1 \end{pmatrix}.$$

COROLLARY 4.3. *The non-zero entries of $M(\{T_{rs}\})$ are all distinct if Γ has no 4-gons. Further, if the ij entry of $M(\{T_{rs}\})$ is non-zero, then so is the ji entry.*

Proof. By Lemma 4.1 a non-zero entry a_{jr} of $M(\{T_{rs}\})$ corresponds to a triple (i, j, r), where $\{j, r\} \in C_2$, $i \in P_1$, and $\{j, i\}$, $\{i, r\} \in C_1$. Thus two entries which are non-zero and equal up to a sign give two triples (i, j, r) and (i', j, r), and so the corresponding vertices form a 4-gon. The last statement follows from the fact that

$$\frac{\partial}{\partial a_{irP}}\left(h(e_{ir})^2(a_{ij})\right) = \pm a_{rj}, \qquad \frac{\partial}{\partial a_{ijP}}\left(h(e_{ij})^2(a_{ir})\right) = \pm a_{jr} \qquad \text{if } \{j,r\} \in C_2. \ ∎$$

5. PRESENTATIONS FOR SOME GROUPS GENERATED BY TRANSVECTIONS

Let $S = \{T_1, \ldots, T_n\}$ be a set of transvections with $T_i = (\psi_i, d_i)$ for $i = 1, \ldots, n$, where $d_1, \ldots, d_n$ is a basis. Assume throughout this section that $M(S) = (m_{ij})$, where either $m_{ij} = 0$ or $m_{ij} = a_{ij}$, where a_{ij} are distinct indeterminates and, further, where m_{ij} is non-zero if and only if m_{ji} is non-zero.

THEOREM 5.1. *The group* $\langle S \rangle$ *has the presentation*

$$\langle T_1, \ldots, T_n | T_i T_j = T_j T_i \text{ if } m_{ij} = 0 \rangle.$$

Proof. To prove this result we will give a normal form for a word in $\langle T_1, \ldots, T_n \rangle$ and then show that any non-trivial word in normal form is a non-trivial element of $\langle S \rangle$. A word $w = T_{\pi(r)}^{\lambda(r)} \ldots T_{\pi(1)}^{\lambda(1)}$ in $\langle S \rangle$ is *T-reduced* if

(1) $\lambda(i)$ is non-zero for $i = 1, \ldots, r$ and $\pi(i) \neq \pi(i+1)$ for $i = 1, \ldots, r-1$; and

(2) If $j > i$, $m_{\pi(i)\pi(j)} = 0$, and $\pi(j) \leq \pi(i)$, then there is $i < k < j$, such that $m_{\pi(k)\pi(j)} \neq 0$.

It is clear that we can always guarantee (1). To see that we can always guarantee (2) we note that if $j > i$, $m_{\pi(i)\pi(j)} = 0$, $\pi(j) \leq \pi(i)$, and $m_{\pi(k)\pi(j)} = 0$ for all $i < k < j$, then using the commutator relations that come from $A(M)$ we can commute $T_{\pi(j)}^{\lambda(j)}$ past all the $T_{\pi(k)}^{\lambda(k)}$'s with $i \leq k < j$. Since $\pi(j) \leq \pi(i)$ this has the effect of either reducing r or strictly reducing the value of the lexicographically ordered sequence $(\pi(r), \ldots, \pi(1))$. Since we cannot continue decreasing the value of this sequence indefinately we see that any word w in $\langle S \rangle$ can be put into T-reduced form using only the commutator relations that come from $A(M)$.

Given a T-reduced word $w = T_{\pi(r)}^{\lambda(r)} \ldots T_{\pi(1)}^{\lambda(1)}$ we define a graph $G(w)$ having vertices $\tau_r, \ldots, \tau_1$ which we think of as lying on a horizontal line from left to right (with τ_r to the left of τ_1). We have an edge in $G(w)$ between vertices τ_i and τ_j whenever $m_{\pi(i)\pi(j)} \neq 0$. An *RL-path* in $G(w)$ is an oriented path in $G(w)$ which always moves from right to left along this line. An *RL-distance function* on the vertices of $G(w)$ is defined as the greatest length of any RL-path between the vertices (or 0 if no such RL-path exists). Given vertices τ_i and τ_j in $G(w)$ with $i < j$, an RL-path from τ_i to τ_j is *maximal* if there is no RL-path in $G(w)$ from τ_i to τ_j of greater length. Given an RL – path p in $G(w)$ there is a corresponding oriented path in $\Gamma(S)$ which we denote by $\mathbf{p}^*$.

Suppose that p_1 and p_2 are maximal RL-paths from τ_i to τ_j in $G(w)$. Then we can split p_1 and p_2 as $p_1 = p_{11} p_{12} \ldots p_{1k}$ and $p_2 = p_{21} p_{22} \ldots p_{2k}$, where $k = 2q - 1$ is odd, p_{1h} and p_{2h} are (perhaps degenerate) RL-paths in $G(w)$ with the same end-points, p_{12h-1} and p_{22h-1} are the same for $h = 1, \ldots, q$, and

$$p_1 \cap p_2 = p_{11} \cup p_{13} \cup \cdots \cup p_{12q-1}.$$

LEMMA 5.2. *For $h = 1, \dots, q-1$ the* RL-*paths $p_{1\,2h}$ and $p_{2\,2h}$ have the same length. Furthermore, $p^*_{1\,2h} \cap p^*_{2\,2h}$ is equal to the two end-points of $p^*_{1\,2h}$.*

Proof. If, for example, there is $h = 1, \dots, q-1$ such that $p_{1\,2h}$ is longer than $p_{2\,2h}$, then, since $p^*_{1\,2h}$ and $p^*_{2\,2h}$ have the same end points, we could replace $p_{2\,2h}$ in p_2 by $p_{1\,2h}$ to obtain an RL-path from τ_i to τ_j with longer length, contradicting the maximality property.

For the second statement, suppose that $p^*_{1\,2h} \cap p^*_{2\,2h}$ is not equal to the two end-points of $p^*_{1\,2h}$. In this case we let v_u be the first common vertex of $p^*_{1\,2h}$ and $p^*_{2\,2h}$ which is not equal to the initial vertex of $p^*_{1\,2h}$. Let r^*_1 and r^*_2 be the initial sub-paths of $p^*_{1\,2h}$ and $p^*_{2\,2h}$ (respectively) which end at v_u. Then r^*_1 and r^*_2 have the same length, else we again contradict the maximality property. Let r_1 and r_2 be the subpaths of $p_{1\,2h}$ and $p_{2\,2h}$ to which r^*_1 and r^*_2 correspond. Then r_1 and r_2 are disjoint except at their initial point. Let x_1 and x_2 be their (distinct) terminal points. Then either x_1 comes before x_2 in $G(w)$ or vice versa. Without loss we may assume that $x_1 = \tau_y$ comes before $x_2 = \tau_z$, i.e., that $y < z$. Then by property (2) above there is $y < t < z$ such that $m_{\pi(y)\pi(t)} \neq 0$ and $m_{\pi(z)\pi(t)} \neq 0$. But now we can find a longer RL-path in $G(w)$ from the initial point of r_1 to $x_2 = \tau_z$, namely, by going along r_1, then to τ_t, and then to τ_z. This contradicts the maximality property again and so the result follows. ∎

Given an RL-path p in $G(w)$ we define the *weight* $\omega(\mathbf{p})$ to be the product

$$2^{\omega(1)}3^{\omega(2)}5^{\omega(3)} \dots P_{n+1}^{\omega(n)},$$

where for $i = 1, \dots, n, \omega(i)$ is the sum of the absolute values of all the $\lambda(h)$'s, where τ_h is a vertex of the path p and $\pi(h) = i$. Here P_{n+1} is the $(n+1)^{\text{th}}$ prime.

COROLLARY 5.3. *Let $1 \leq i < j \leq r$. Then among all maximal* RL-*paths from τ_i to τ_j there is a unique such* RL-*path of maximal weight.*

Proof. Let p_1 be an RL-path from τ_i to τ_j of maximal weight among all such maximal paths. Let p_1 be a different maximal RL-path from τ_i to τ_j. We need to show that p_2 has strictly smaller weight than p_1. Now we can decompose p_1 and p_2 as we did in the above. Then the weights of the common subpaths $p_{1\,2h-1}$ and $p_{2\,2h-1}$ are clearly equal for $h = 1, \dots, q$. Now by Lemma 5.2, $p^*_{1\,2h} \cap p^*_{2\,2h}$ is equal to the two end-points of $p^*_{1\,2h}$ for $h = 1, \dots, q$ and so the interior vertices of these paths are distinct. Now the contributions to $\omega(p)$ that the paths $p_{1\,2h}$ and $p_{2\,2h}$ make are clearly distinct, one being smaller than the other. Since p_1 has maximal weight we see that the contribution made by $p_{1\,2h}$ must be the bigger of the two and so the result follows. ∎

Given an RL-path p in $G(w)$, passing in order through the vertices $\tau_b, \tau_c, \tau_d, \dots, \tau_g, \tau_h$ of $G(w)$ and $1 \le u \le n$, we let

$$p_u^{\#} = a_{\pi(b)\pi(c)} a_{\pi(c)\pi(d)} \cdots a_{\pi(g)\pi(h)} a_{\pi(h)\pi(u)}$$

and

$$\varepsilon(p) = \lambda(b)\lambda(c)\dots\lambda(h).$$

A vertex τ_u of $G(w)$ is an *L-end vertex* if there is no $b > u$ such that $m_{\pi(b)\pi(u)} \neq 0$. For $1 \le i \le n$ let $e_i = (0,\dots,0,1,0,\dots,0)^t$ with the 1 in the i^{th} position (here t denotes transpose).

LEMMA 5.4. *Let $w = T_{\pi(r)}^{\lambda(r)} \dots T_{\pi(1)}^{(\lambda(1)}$ be a T-reduced word. Then for $1 \le u \le n$ the i^{th} coordinate of $w(e_u)$ is equal to*

$$\delta_{iu} + \sum_p \varepsilon(p) p_u^{\#},$$

where the sum is over all maximal RL-paths p in $G(w)$ which end at a vertex τ_h, where $\pi(h) = i$.

Proof. This is by induction on the length r of a T-reduced word, $w = T_{\pi(r)}^{\varepsilon(r)} \dots T_{\pi(1)}^{\varepsilon(1)}$. We can do this since it easily follows from (1) and (2) above that any initial subword $w' = T_{\pi(s)}^{\varepsilon(s)} \dots T_{\pi(1)}^{\varepsilon(1)}$ of w, where $s < r$ is also T-reduced. If $r = 1$, then

$$w(e_u) = e_u + \left(0,\dots,0,\lambda(1)m_{\pi(1)u},0,\dots,0\right)^t$$

where $\lambda(1)m_{\pi(1)u}$ is in the $\pi(1)$ position and so it has the correct form.

Now assume that the lemma is true for $w'(e_u) = T_{\pi(r-1)}^{\lambda(r-1)} \dots T_{\pi(1)}^{\lambda(1)}(e_u) = (\alpha_1,\dots,\alpha_n)^t$. Then

$$\begin{aligned} w(e_u) = T_{\pi(r)}^{\lambda(r)}(\alpha_1,\dots,\alpha_n)^t = \big(\alpha_1,\dots,\alpha_{\pi(r)-1},\alpha_{\pi(r)} &+ \lambda(r)m_{\pi(r)1}\alpha_1 \\ &+ \cdots + \lambda(r)m_{\pi(r)n}\alpha_n, \alpha_{\pi(r)+1},\dots,\alpha_n\big)^t. \end{aligned}$$

Now if there is any RL-path p in $G(w)$ that is not in $G(w')$, then it must involve the vertex τ_r which would then be an L-end vertex of this path. Thus p must be of the form $p = \tau_r p'$, where p' is a path in $G(w')$ which has L-end vertex τ_s, where $s < r$ and $m_{\pi(r)\pi(s)} \neq 0$. This thus corresponds to a summand $\lambda(r)m_{\pi(r)\pi(s)}\alpha_s$ which is not in $w'(e_u)$. Conversely, any such summand corresponds to a path p in $G(w)$ of the form $p = \tau_r p'$, where p' is a path in $G(w')$, i.e., to a path p which has τ_r as its L-end vertex. Induction now gives the result.

By Lemmas 5.3 and 5.4 we see that if w is a non-trivial T-reduced word, then each coordinate of $w(e_u)$ has a unique monomial of highest degree and highest weight with non-zero coefficient. Thus w is not the identity and so Theorem 5.1 follows. ▌

Since the degree and weight functions would be the same the proof of Theorem 5.1 can easily be adapted to prove

COROLLARY 5.5. *If $M(S) = (m_{ij})$, where either $m_{ij} = 0$ or $m_{ij} = a_{ij}$ with a_{ij} distinct indeterminates, except that for $1 \le i < j \le n$ we have $a_{ij} = b_{ij}a_{ji}$, where $b_{ij} \in \mathbb{C} \setminus \{0\}$ and, further, where m_{ij} is non-zero if and only if m_{ji} is non-zero, then the group $\langle S \rangle$ has the presentation*

$$\langle T_1, \dots, T_n | T_iT_j = T_jT_i \text{ if } m_{ij} = 0 \rangle.$$

Remark. The linear representation of $A(\Gamma)$ described above does not give the correct kind of representation for Γ equal to the 4-gon graph with vertices $v_1, \dots, v_4$, where v_i and v_j are adjacent if and only if $i \equiv j \pm 1 \bmod 4$ as in Example 4.2. A relator is

$$T_{34}T_{12}^{-1}T_{41}T_{12}T_{34}^{-1}T_{23}^{-1}T_{34}T_{12}^{-1}T_{41}^{-1}T_{12}T_{34}^{-1}T_{23}.$$

6. A GENERALISATION OF TITS' CONJECTURE FOR BRAID GROUPS

Let P_n be the subgroup of pure braids in B_n; i.e., P_n is the normal closure of σ_1^2. Then P_n is generated by elements

$$A_{ij} = (\sigma_{j-1}\sigma_{j-2} \dots \sigma_{i+1})\sigma_i(\sigma_{j-1}\sigma_{j-2} \dots \sigma_{i+1})^{-1}$$

for $1 \le i < j \le n$ [Bi, p. 20]. We let

$$B_{ij} = \left(\sigma_{j-1}^{-1}\sigma_{j-2}^{-1} \dots \sigma_{i+1}^{-1}\right)\sigma_i\left(\sigma_{j-1}^{-1}\sigma_{j-2}^{-1} \dots \sigma_{i+1}^{-1}\right)^{-1}.$$

Then A_{ij} and B_{ij} are Dehn twists about curves a_{ij} and b_{ij} in the n-puntured disc D_n shown in Fig. 4.

Let $O_n = \{A_{ij}, B_{ij} | 1 \le i < j \le n\}$. Then for any subset $O' \subset O_n$ we let $\Gamma(O')$ be the graph with vertices $v_1, \dots, v_n$ and an edge from v_r to v_s for every A_{rs} or B_{rs} in O'. There is a natural homeomorphism $\eta \colon \Gamma(O') \to D_n$, where vertices are sent to the punctures of D_n and an edge corresponding to a generators A_{rs} (respectively B_{rs}) is drawn from v_r to v_s below (respectively above) the horizontal line of punctures in Fig. 4.

THEOREM 6.1. *If $O' \subset O_n$ and $\Gamma(O')$ is bipartite, has no 2-gons or 4-gons and the map $\eta \colon \Gamma(O') \to D_n$ is an embedding, then $\langle O' \rangle$ is a graph group.*

Proof. By the results of Section 4 the standard representation $B_n \to \mathrm{Aut}(F_n)$ [Bi, p. 30] gives an action of $\langle O' \rangle$ on $\mathbb{Z}[a_{ij}]$ since the image of B_n

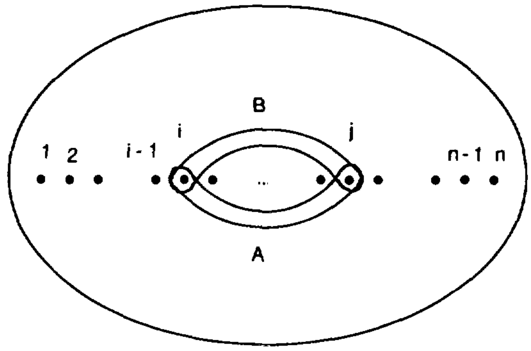

FIGURE 4

lies in $\hat{\mathrm{H}}(n)$. We also have a representation $A(\Gamma(O')) \to \mathrm{Aut}(F_n)$. One checks that if e is an edge of $A(\Gamma(O'))$, corresponding to the generator A_{rs} (respectively B_{rs}), then the actions of $\sigma(e)^2$ and A_{rs} (respectively B_{rs}) on $\mathbb{Z}[a_{ij}]$ are the same. By Theorem 1.1 the representation of $A(\Gamma(O'))$ is a graph group and so the result follows. ∎

Example 6.2. Let

$$O' = \{A_{ii+1} | 0 < i < n\} \cup \{A_{ij} | i < j,\, j - 1 = k5^{r-1},\, j - i = 5^{r-1}\}$$
$$\cup \{B_{ij} | i < j,\, j - 3 = k5^{r-1},\, j - i = 5^{r-1}\}.$$

Then O' satisfies the conditions of 6.1. The set O' has

$$n - 1 + \left[\frac{n-1}{5}\right] + \left[\frac{n-1}{5^2}\right] + \cdots + \left[\frac{n-3}{5}\right] + \left[\frac{n-3}{5^2}\right] + \cdots$$

elements in it, where $[a]$ is the largest integer not greater than a.

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